Lecture 16: Recall: Successive overrelaxation method (SOR)

Consider the iterative scheme:

(Suppose $A = L + D + U$)

$$L\vec{x}^{k+1} + D\vec{y}^{k+1} + U\vec{x}^{k} = \vec{b} \quad \text{—— } (*)$$

$$\vec{x}^{k+1} = \vec{x}^{k} + \underset{\mathbb{R}}{\omega}\,(\vec{y}^{k+1} - \vec{x}^{k}) \quad \text{—— } (**)$$

$$\Longleftrightarrow \quad \vec{y}^{k+1} = \frac{1}{\omega}\left(\vec{x}^{k+1} + (\omega-1)\vec{x}^{k}\right)$$

Putting $(**)$ into $(*)$:

$$\left(L + \frac{1}{\omega}D\right)\vec{x}^{k+1} + \frac{1}{\omega}\left(\omega U + (\omega-1)D\right)\vec{x}^{k} = \vec{b}$$

or

$$\underbrace{\left(L + \frac{1}{\omega}D\right)}_{N}\vec{x}^{k+1} = \underbrace{\left(\frac{1}{\omega}D - (D+U)\right)}_{P}\vec{x}^{k} + \vec{b}$$

Remark:

$$M_{SOR} = N_{SOR}^{-1} P_{SOR}$$

$$= \left(L + \frac{1}{\omega} D\right)^{-1} \left(\frac{1}{\omega} D - (D+U)\right)$$

$$= \left(\frac{1}{\omega}(D+\omega L)\right)^{-1} \left(\frac{1}{\omega}(D - \omega(D+U))\right)$$

$$= \cancel{\omega}\,(D+\omega L)^{-1} \cancel{\left(\frac{1}{\omega}\right)} \left((1-\omega)D - \omega U\right)$$

$$= (D+\omega L)^{-1}\left((1-\omega)D - \omega U\right)$$

Theorem: If A is strictly diagonally dominant (SDD), then SOR converges if $0 < \omega \leq 1$.

Proof: We need to show $\rho(M_{SOR}) < 1$ if $0 < \omega \leq 1$.

We'll show it by contradiction.

Suppose $\exists$ eigenvalue λ such that $|\lambda| \geq 1$. Then:

$$\det(\lambda I - M_{SOR}) = 0$$

$$\therefore \det\left(\lambda I - (D+\omega L)^{-1}((1-\omega)D - \omega U)\right) = 0$$

$$\Rightarrow \det\left(\underbrace{\lambda}_{\neq 0}(D+\omega L)^{-1}\underbrace{\left((D+\omega L) - \frac{1}{\lambda}((1-\omega)D - \omega U)\right)}_{= C}\right) = 0$$

$\Rightarrow \det(C) = 0$ ($\because \lambda \neq 0$, $(D+\omega L)$ is invertible)

Note: $\omega\left(1-\frac{1}{|\lambda|}\right) \le \left(1-\frac{1}{|\lambda|}\right) \Rightarrow \left(1-\frac{1}{|\lambda|}(1-\omega)\right) \ge \omega$

Now,

$$|C_{ii}| = \left|1-\frac{1}{\lambda}(1-\omega)\right| |a_{ii}| \ge \left[1-\frac{1}{|\lambda|}(1-\omega)\right] |a_{ii}|$$

$$\ge \omega |a_{ii}| > \omega \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \ge \underbrace{\omega \sum_{j=1}^{i-1} |a_{ij}| + \frac{\omega}{|\lambda|} \sum_{j=i+1}^{n} |a_{ij}|}_{= \sum_{\substack{j=1 \\ j\neq i}}^{n} |C_{ij}|}$$

$$|C_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |C_{ij}| \quad \text{for all } i$$

$\therefore$ C is SDD $\Rightarrow$ C is non-singular!!

$\Rightarrow \det(C) \neq 0$

Contradiction.

$$\boxed{\underbrace{\left((D+\omega L) - \frac{1}{\lambda}\left((1-\omega)D - \omega U\right)\right)}_{= C}}$$

Optimal parameter ω_{opt} for SOR method

Definition: Consider the system $A\vec{x} = \vec{b}$. Let $A = \overset{\text{lower}}{L} + \overset{\text{diagonal}}{D} + \overset{\text{upper}}{U}$.

If the eigenvalues of : $\alpha D^{-1} L + \frac{1}{\alpha} D^{-1} U$ $(\alpha \neq 0)$ are independent of α. Then, the matrix A is said to be consistently ordered.

Example of consistently ordered matrices

1. Tridiagonal matrix :

$$\begin{pmatrix} \lambda_1 & * & & 0 \\ * & \lambda_2 & * & \\ & \ddots & \ddots & * \\ 0 & & * & \lambda_n \end{pmatrix}$$

2. Block tridiagonal matrix :

$$\begin{pmatrix} \boxed{D_1} & T_{12} & & 0 \\ T_{21} & \boxed{D_2} & T_{23} & \\ & \ddots & \ddots & \ddots \\ 0 & & & \boxed{D_p} \end{pmatrix}$$

where D_i = diagonal matrix

Example: Consider $A\vec{x} = \begin{pmatrix} 10 & 1 \\ 1 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}$

Then: $\alpha D^{-1}L + \frac{1}{\alpha} D^{-1}U = \begin{pmatrix} 0 & -1/10\alpha \\ \frac{-\alpha}{10} & 0 \end{pmatrix}$

$\therefore$ Char. poly $= \lambda^2 - \left(\frac{-1}{10\cancel{\alpha}}\right)\left(\frac{-\cancel{\alpha}}{10}\right) = 0 \Rightarrow \lambda^2 - \frac{1}{100} = 0$

(independent of α)

$\therefore$ A is consistently ordered.

Theorem: [D. Young] Assume:

1. $0 < \omega < 2$
2. $M_J = N_J^{-1} P_J$ has only real eigenvalues
3. $\beta \overset{def}{=} \rho(M_J) < 1$
4. A is consistently ordered.

Then: $\rho(M_{SOR}) < 1$

Also, the optimal parameter ω_{opt} for fastest convergence

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}} \quad \text{and}$$

$$\rho(M_{SOR,\omega_{opt}}) = \omega_{opt} - 1$$